Superconductors with Topological Order and their Realization in Josephson Junction Arrays

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February 21, 2007

We will describe a new superconductivity mechanism, proposed by the authors in [1], which is based on a topologically ordered ground state rather than on the usual Landau mechanism of spontaneous symmetry breaking. Contrary to anyon superconductivity it works in any dimension and it preserves P-and T-invariance. In particular we will discuss the low-energy effective field theory, what would be the Landau-Ginzburg formulation for conventional superconductors.

1 Introduction

For many years the theory of phase transitions was entirely understood in terms of the Landau-Ginzburg theory, based on symmetry breaking, order parameters and on the mathematical framework of group theory. With the discovery of the fractional quantum Hall liquids [2], that are incompressible and exist only at some ”magical” filling fractions, it was understood that the internal order characterizing these states is a new type of order, different from any other known type of order. This new type of order is the topological order [3], a particular type of quantum order.

Quantum order describes the zero-temperature properties of a quantum ground state, and characterizes universality classes of quantum states, described by *complex* ground state wave-functions [3]. Quantum phase transition are characterized by changes in the quantum entanglement properties of these complex ground state wave-functions. Topological order is a special type of quantum order whose hallmarks are the presence of a gap for all excitations (incompressibility) and the degeneracy of the ground state on manifolds with non-trivial topology [3]. In the case of the fractional quantum Hall effect, which is a (2+1)-dimensional system, another hallmark is the presence of excitations with fractional spin and statistics, called anyons [4]. The long-distance properties of these topological fluids can be explained by an infinite-dimensional $W_{1+\infty}$ dynamical symmetry [5], and are described by effective Chern-Simons field theories [6] with compact gauge group, which break P-and T-invariance.

The topologically ordered superconductors we propose have a long-distance hydrodynamic action which can be entirely formulated in terms of generalized compact gauge fields, the dominant term being the topological BF action [7]. The BF theory has many applications, from 2D gravity [8] to the mathematical characterization of embedded manifolds [9]. It turns out that it is also the general model that describes the long-distance behaviour of systems that exhibit P- and T-preserving topological order in any dimension. It reduces to a doubled Chern-Simons model in (2+1)-dimensions [10]. The existence of such non-conventional superconductors is also supported by purely algebraic considerations [11].

In [12], we have proven that planar Josephson junction arrays (JJA) can be exactly mapped onto an Abelian gauge theory with a mixed Chern-Simons term (BF-model): JJA provide thus a first concrete example of topological superconductors. The Abelian gauge theory exactly reproduces the phase diagram of JJA and the insulator/superconductor quantum phase transition at $T = 0$ [13]. JJA have also been recently considered by several other authors [14, 15], as controllable devices which exhibit topological order.

The idea that gauge fields can be used to model the long distance behaviour of condensed matter systems was originally proposed in [17], and particularly exploited for planar systems [18]. In a nutshell, the idea is that charge fluctuations around a given ground state are decribed by a conserved current j^μ, which in (2+1) dimensions can be represented in terms of a gauge field B_μ according to $j^\mu \propto \epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu$. For a wide class of systems the effective action governing the dynamics of the charge fluctuations is quadratic in the gauge fields B_μ at long distances [17] . Clearly this effective action must also be gauge invariant, reflecting the original gauge invariance of the current j^μ: one obtains thus an effective gauge theory at long distances (which is not necessarily relativistic). The ground states of a wide class of planar condensed matter systems [19] can be classified according to the lowest derivative terms appearing in their effective gauge theories at long distances. This way, the Chern-Simons term describes incompressible quantum fluids (quantum Hall states) and chiral spin liquids [20] while the Maxwell term describes a (2-dim.) superfluid (superconductor).

The case of JJA is different in the sense that the gauge theory description is an exact mapping and not only an effective theory [12]: JJA can be *exactly mapped* onto an Abelian gauge theory with two gauge fields describing a current of charges and a current of vortices coupled by a mixed Chern-Simons term:

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2\pi}A_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu - \frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu} \ , \tag{1}$$

Global superconductivity in planar JJA is thus the simplest example of the new mechanism of superconductivity we propose.

We argue [21] that a less trivial example are frustrated JJA: these may support a topologically ordered superconducting ground state characterized by a non-trivial ground state degeneracy on the torus. These superconducting quantum fluids provide explicit examples of systems in which superconductivity arises from a topological mechanism rather than from the usual Landau-Ginzburg mechanism.

The paper is organized as follows. In Section 2 we will review the definition of the Chern-Simons model on the lattice. In Section 3 we show how planar JJA can be exactly mapped onto a gauge theory with two Maxwell fields coupled by a mixed, periodic, Chern-Simons term. Section 4 is devoted to the study of the phase diagram, showing how the superconductor/insulating transition of the JJA is reproduced by the gauge theory. In Section 5 we present the general theory of topological superconductors in any space-time dimension. Section 6 is, instead, devoted to the study of the role of frustration.

2 Lattice Chern-Simons model

Our model (1) can be rewritten in terms of the dual field strengths

$$\begin{aligned} F^\mu &\equiv \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}\ , \qquad && F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\ , \\ f^\mu &\equiv \frac{1}{2}\epsilon^{\mu\alpha\beta}f_{\alpha\beta}\ , \qquad && f_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu\ , \end{aligned} \tag{2}$$

as follows (throughout this paper we use units such that $c = 1$ and $\hbar = 1$.)

$$\mathcal{L}_{CS} = -\frac{1}{2e^2}\left(\frac{1}{\eta}F_0F^0 + F_iF^i\right) + \frac{\kappa}{2\pi}A_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha B_\nu - \frac{1}{2g^2}\left(\frac{1}{\eta}f_0f^0 + f_if^i\right)\ . \tag{3}$$

For later convenience we have introduced a magnetic permeability η, equal for the two gauge fields. The coupling constants e^2 and g^2 have dimension mass, whereas the coefficient κ of the mixed Chern-Simons term is dimensionless. Note that we take B_μ to represent a pseudovector gauge field, so that the mixed Chern-Simons term does not break the discrete symmetries of parity and time reversal.

The action corresponding to (3) is separately invariant under the two Abelian gauge transformations

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\lambda\ , \\ B_\mu &\to B_\mu + \partial_\mu\omega\ , \end{aligned} \tag{4}$$

with gauge groups R_A and R_B, respectively. Moreover, the action is also invariant under the *duality transformation*

$$\begin{aligned} A_\mu &\leftrightarrow B_\mu\ , \\ e &\leftrightarrow g\ , \end{aligned} \tag{5}$$

so that the model is *self-dual*.

The Lagrangian (3) can be easily diagonalized by the linear transformation

$$\begin{aligned} A_\mu &= \sqrt{\frac{e}{g}}\left(a_\mu + b_\mu\right)\ , \\ B_\mu &= \sqrt{\frac{g}{e}}\left(a_\mu - b_\mu\right)\ . \end{aligned} \tag{6}$$

In terms of these new variables the model (3) describes a free theory,

$$\mathcal{L}_{CS} = -\frac{1}{eg}\left(\frac{1}{\eta}G_0G^0 + G_iG^i\right) + \frac{\kappa}{2\pi}a_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha a_\nu - \frac{1}{eg}\left(\frac{1}{\eta}g_0g^0 + g_ig^i\right) - \frac{\kappa}{2\pi}b_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha b_\nu \ , \tag{7}$$

where G^μ and g^μ are the dual field strengths for the new gauge fields a_μ and b_μ, respectively. This Lagrangian describes a doublet of excitations with topological mass [22]

$$m = \frac{|\kappa| eg}{2\pi} \ , \tag{8}$$

and spectrum

$$E(\mathbf{q}) = \sqrt{m^2 + \frac{1}{\eta}|\mathbf{q}|^2} \ . \tag{9}$$

In the following we shall formulate a Euclidean lattice version of the above Chern-Simons model. To this end we introduce a three-dimensional rectangular lattice with lattice spacings l_μ in the three directions. In particular we shall take the lattice spacings $l_1 = l_2 \equiv l$ and identify l_0 with the spacing in the Euclidean time direction. Lattice sites are denoted by the three-dimensional vector x; the gauge fields $A_\mu(x)$ and $B_\mu(x)$ are associated with the links (x, μ) between the sites x and $x + \hat{\mu}$, where $\hat{\mu}$ denotes a unit vector in direction μ on the lattice.

On the lattice we introduce the following forward and backward derivatives and shift operators:

$$\begin{aligned} d_\mu f(x) &\equiv \frac{f(x + l_\mu\hat{\mu}) - f(x)}{l_\mu} \ , \qquad & S_\mu f(x) \equiv f(x + l_\mu\hat{\mu}) \ , \\ \hat{d}_\mu f(x) &\equiv \frac{f(x) - f(x - l_\mu\hat{\mu})}{l_\mu} \ , \qquad & \hat{S}_\mu f(x) \equiv f(x - l_\mu\hat{\mu}) \ . \end{aligned} \tag{10}$$

Summation by parts on the lattice interchanges both the two derivatives (with a minus sign) and the two shift operators; gauge transformations are defined using the forward lattice derivative. Corresponding to the two derivatives in 10 , we can define also two lattice analogues of the Chern-Simons operators $\epsilon_{\mu\alpha\nu}\partial_\alpha$ [23] [12] :

$$k_{\mu\nu} \equiv S_\mu \epsilon_{\mu\alpha\nu} d_\alpha \ , \qquad \hat{k}_{\mu\nu} \equiv \epsilon_{\mu\alpha\nu}\hat{d}_\alpha \hat{S}_\nu \ , \tag{11}$$

where no summation is implied over equal indices μ and ν. Summation by parts on the lattice interchanges also these two operators (without an extra minus sign). The operators 11 are both local and gauge invariant, in the sense that

$$k_{\mu\nu} d_\nu = \hat{d}_\mu k_{\mu\nu} = 0 \ , \qquad \hat{k}_{\mu\nu} d_\nu = \hat{d}_\mu \hat{k}_{\mu\nu} = 0 \ , \tag{12}$$

and their product reproduces the relativistic, Euclidean lattice Maxwell operator:

$$k_{\mu\alpha}\hat{k}_{\alpha\nu} = \hat{k}_{\mu\alpha}k_{\alpha\nu} = -\delta_{\mu\nu}\nabla^2 + d_\mu \hat{d}_\nu \ , \tag{13}$$

where $\nabla^2 \equiv \hat{d}_\mu d_\mu$ is the three-dimensional Laplace operator. Using $k_{\mu\nu}$ we can also define the lattice dual field strengths as

$$\begin{aligned} F_\mu &\equiv \hat{k}_{\mu\nu} A_\nu \ , \\ f_\mu &\equiv k_{\mu\nu} B_\nu \ . \end{aligned} \tag{14}$$

The identity (13) then tells us that we can simply write the relativistic, Euclidean lattice Maxwell terms as $\sum_x F_\mu F_\mu$ and $\sum_x f_\mu f_\mu$.

Using all these definitions we can now write the Euclidean lattice partition function of our model (3) as follows:

$$\begin{aligned} Z_{CS} &= \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp(-S_{CS}) \ , \\ S_{CS} &= \sum_x \frac{l_0 l^2}{2e^2} \left(\frac{1}{\eta} F_0 F_0 + F_i F_i \right) - i \frac{l_0 l^2 \kappa}{2\pi} A_\mu k_{\mu\nu} B_\nu + \frac{l_0 l^2}{2g^2} \left(\frac{1}{\eta} f_0 f_0 + f_i f_i \right) \ , \end{aligned} \tag{15}$$

where we have introduced the notation $\mathcal{D}A_\mu \equiv \prod_{(x,\mu)} dA_\mu(x)$ and gauge fixing is understood.

For later convenience we introduce also the finite difference operators

$$\Delta_\mu \equiv l_\mu d_\mu \ , \qquad \hat{\Delta}_\mu \equiv l_\mu \hat{d}_\mu \ , \tag{16}$$

where no summation over equal indices is implied. Correspondingly, we introduce also the finite difference analogue of the operators $k_{\mu\nu}$ and $\hat{k}_{\mu\nu}$:

$$K_{\mu\nu} \equiv S_\mu \epsilon_{\mu\alpha\nu} \Delta_\alpha \ , \qquad \hat{K}_{\mu\nu} \equiv \epsilon_{\mu\alpha\nu} \hat{\Delta}_\alpha \hat{S}_\nu \ . \tag{17}$$

These satisfy equations analogous to (12) and (13) with all derivatives substituted by finite differences.

3 Josephson junction arrays

JJA we analize are quadratic, planar arrays of spacing l of superconducting islands with nearest neighbours Josephson couplings of strength E_J. Each island has a capacitance C_0 to the ground; moreover there are also nearest neighbours capacitances C. The Hamiltonian characterizing such systems is thus given by

$$H = \sum_{\mathbf{x}} \frac{C_0}{2} V_{\mathbf{x}} + \sum_{<\mathbf{xy}>} \left(\frac{C}{2} \left(V_{\mathbf{y}} - V_{\mathbf{x}} \right)^2 + E_J \left(1 - \cos N \left(\Phi_{\mathbf{y}} - \Phi_{\mathbf{x}} \right) \right) \right) \ , \tag{18}$$

where boldface characters denote the sites of the two-dimensional array, $< \mathbf{xy} >$ indicates nearest neighbours, $V_{\mathbf{x}}$ is the electric potential of the island at $\mathbf{x}$ and $\Phi_{\mathbf{x}}$ the phase of its order parameter. For generality we allow for any integer N in the Josephson coupling, so that the phase has periodicity $2\pi/N$: obviously $N = 2$ for the real systems.

With the notation introduced in the previous section the Hamiltonian (18) can be rewritten as

$$H = \sum_{\mathbf{x}} \frac{1}{2} V \left(C_0 - C\Delta\right) V + \sum_{\mathbf{x},i} E_J \left(1 - \cos N \left(\Delta_i \Phi\right)\right) \ , \tag{19}$$

where $\Delta \equiv \hat{\Delta}_i \Delta_i$ is the two-dimensional finite difference Laplacian and we have omitted the explicit location indeces on the variables V and Φ.

The phases $\Phi_{\mathbf{x}}$ are quantum-mechanically conjugated to the charges $Q_{\mathbf{x}}$ on the islands: these are quantized in integer multiples of N (Cooper pairs for $N = 2$):

$$Q = q_e N p_0 \ , \qquad\qquad p_0 \in Z \tag{20}$$

where q_e is the electron charge. The Hamiltonian (19) can be expressed in terms of charges and phases by noting that the electric potentials $V_{\mathbf{x}}$ are determined by the charges $Q_{\mathbf{x}}$ via a discrete version of Poisson's equation:

$$\left(C_0 - C\Delta\right) V_{\mathbf{x}} = Q_{\mathbf{x}} \ . \tag{21}$$

Using this in (19) we get

$$H = \sum_{\mathbf{x}} N^2 E_C \ p_0 \frac{1}{\frac{C_0}{C} - \Delta} p_0 + \sum_{\mathbf{x},i} E_J \left(1 - \cos N \left(\Delta_i \Phi\right)\right) \ , \tag{22}$$

where $E_C \equiv q_e^2/2C$. The integer charges p_0 interact via a two-dimensional Yukawa potential of mass $\sqrt{C_0/C}/l$. In the nearest-neighbours capacitance limit $C \gg C_0$, which is accessible experimentally, this becomes essentially a two-dimensional Coulomb law. ¿From now on we shall consider the limiting case $C_0 = 0$. In this case the *charging energy* E_C and the *Josephson coupling* E_J are the two relevant energy scales in the problem. These two massive parameters can also be traded for one massive parameter $\sqrt{2N^2 E_C E_J}$, which represents the *Josephson plasma frequency* and one massless parameter E_J/E_C.

The zero-temperature partition function of the Josephson junction array admits a (phase-space) path-integral representation [24]. Since the variables p_0 are integers, the imaginary-time integration has to be performed stepwise; we introduce therefore a lattice spacing l_0 also in the imaginary-time direction. This has to be just smaller of the typical time scale on which the integers p_0 vary, in the present case the inverse of the Josephson plasma frequency: $l_0 \leq O\left(1/\sqrt{2N^2 E_C E_J}\right)$. We thus get the following partition function:

$$\begin{aligned} Z \quad &= \sum_{\{p_0\}} \int_{-\pi/N}^{+\pi/N} \mathcal{D}\Phi \ \exp(-S) \ , \\ S \quad &= \sum_{x} -iN \ p_0 \Delta_0 \Phi + N^2 E_C l_0 \ p_0 \frac{1}{-\Delta} p_0 + \sum_{x,i} l_0 E_J \left(1 - \cos N \left(\Delta_i \Phi\right)\right) \ , \end{aligned} \tag{23}$$

where now the sum in the action S extends over the three-dimensional lattice with spacing l_0 in the imaginary time direction and l in the spatial directions.

In the next step we introduce vortex degrees of freedom by replacing the Josephson term by its Villain form:

$$\begin{aligned} Z &= \sum_{\substack{\{p_0\}\\ \{v_i\}}} \int_{-\pi/N}^{+\pi/N} \mathcal{D}\Phi \exp(-S) \ , \\ S &= \sum_x -iN \ p_0 \Delta_0 \Phi + N^2 E_C l_0 \ p_0 \frac{1}{-\Delta} p_0 + N^2 l_0 \frac{E_J}{2} \left(\Delta_i \Phi + \frac{2\pi}{N} v_i \right)^2 \ . \end{aligned} \quad (24)$$

Strictly speaking, this substitution is valid only for $l_0 E_J \gg 1$; however the Villain approximation retains all most relevant features of the Josephson coupling for the whole range of values of the coupling E_J and therefore we shall henceforth adopt it.

We now represent the Villain term as a Gaussian integral over real variables p_i and we transform also p_0 to a real variable by introducing new integers v_0 via the Poisson summation formula

$$\sum_{k=-\infty}^{k=+\infty} \exp(i2\pi kz) = \sum_{n=-\infty}^{n=+\infty} \delta(z-n) \ . \quad (25)$$

By grouping together the real and integer p and v variables into three-vectors p_μ and v_μ, $\mu = 0, 1, 2$ we can write the partition function as

$$\begin{aligned} Z &= \sum_{\{v_\mu\}} \int \mathcal{D}p_\mu \int_{-\pi/N}^{+\pi/N} \mathcal{D}\Phi \exp(-S) \ , \\ S &= \sum_x -iN p_\mu \left(\Delta_\mu \Phi + \frac{2\pi}{N} v_\mu \right) + N^2 E_C l_0 \ p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2 l_0 E_J} \ . \end{aligned} \quad (26)$$

Following [25] we use the longitudinal part of the integer vector field v_μ to shift the integration domain of Φ. To this end we decompose v_μ as follows:

$$v_\mu = \Delta_\mu m + \Delta_\mu \alpha + K_{\mu\nu} \psi_\nu \ , \quad (27)$$

where $m \in Z$, $|\alpha| < 1$ and $K_{\mu\nu}$ defined in (17) . Here the vectors ψ_μ are not integer, but they are nonetheless restricted by the fact that the combinations $q_\mu \equiv \hat{K}_{\mu\nu} v_\nu = \hat{K}_{\mu\alpha} K_{\alpha\nu} \psi_\nu$ must be integers. The original sum over the three independent integers $\{v_\mu\}$ can thus be traded for a sum over the four integers $\{m, q_\mu\}$ subject to the constraint $\hat{\Delta}_\mu q_\mu = 0$. The sum over the integers $\{m\}$ can then be used to shift the Φ integration domain from $[-\pi/N, +\pi/N)$ to $(-\infty, +\infty)$. The integration over Φ is now trivial and enforces the constraint $\hat{\Delta}_\mu p_\mu = 0$:

$$\begin{aligned} Z &= \sum_{\{q_\mu\}} \delta_{\hat{\Delta}_\mu q_\mu, 0} \int \mathcal{D}p_\mu \ \delta\left(\hat{\Delta}_\mu p_\mu \right) \ \exp(-S) \ , \\ S &= \sum_x -i2\pi \ p_\mu K_{\mu\nu} \psi_\nu + N^2 l_0 E_C \ p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2 l_0 E_J} \ . \end{aligned} \quad (28)$$

We now solve the two constraints by introducing a real gauge field b_μ and an integer gauge field a_μ:

$$\begin{aligned} p_\mu &\equiv K_{\mu\nu} b_\nu \ , & b_\mu \in R \ , \\ q_\mu &\equiv \hat{K}_{\mu\nu} a_\nu \ , & a_\mu \in Z \ . \end{aligned} \tag{29}$$

By inserting the first of these two equations and by summing by parts, the first term in the action (28) reduces to $\sum_x -i2\pi b_\mu q_\mu$. By inserting the second of the above equations and by summing by parts again, this term of the action finally reduces to the mixed Chern-Simons coupling $\sum_x -i2\pi \ a_\mu K_{\mu\nu} b_\nu$. Using the Poisson formula (25) we can finally make a_μ also real at the expense of introducing a set of integer link variables $\{Q_\mu\}$ satisfying the constraint $\hat{\Delta}_\mu Q_\mu$, which guarantees gauge invariance:

$$\begin{aligned} Z &= \sum_{\{Q_\mu\}} \int \mathcal{D}a_\mu \int \mathcal{D}b_\mu \exp(-S) \ , \\ S &= \sum_x -i2\pi \ a_\mu K_{\mu\nu} b_\nu + N^2 l_0 E_C \ p_0 \frac{1}{-\Delta} p_0 + \frac{p_i^2}{2 l_0 E_J} + i2\pi a_\mu Q_\mu \ . \end{aligned} \tag{30}$$

In this representation $K_{\mu\nu} b_\nu$ represents the conserved three-current of charges, while $\hat{K}_{\mu\nu} a_\nu$ represents the conserved three-current of vortices. Note that, actually, both these conserved currents are integers (the factors of N are explicit): indeed, the summation over $\{Q_\mu\}$ makes a_μ (and therefore also $\hat{K}_{\mu\nu} a_\nu$) an integer, and then the summation over $\{a_\mu\}$ makes $K_{\mu\nu} b_\nu$ an integer. The third term in the action (30) contains two parts: the longitudinal part $\left(p_i^L\right)^2$ describes the Josephson currents and represents a kinetic term for the charges; the transverse part $\left(p_i^T\right)^2$ can be rewritten as a Coulomb interaction term for the vortex density q_0 by solving the Gauss law enforced by the Lagrange multiplier b_0.

The partition function (30) displays a high degree of symmetry between the charge and the vortex degrees of freedom. The only term which breaks this symmetry (apart from the integers Q_μ) is encoded in the kinetic term for the charges (Josephson currents). This near-duality between charges and vortices has already been often invoked in the literature [16] to explain the experimental quantum phase diagram at very low temperatures. Here we introduce what we call the *self-dual approximation* of Josephson junction arrays. This consists in adding to the action in (30) a bare kinetic term for the vortices (note that such a kinetic term is anyhow induced by integrating out the charge degrees of freedom) and combining this with the Coulomb term for the charges into $\sum_x \frac{\pi^2}{N^2 l_0 E_C} q_i^2$. The coefficient is chosen so that the transverse part of this term reproduces exactly the Coulomb term for the charges upon solving the Gauss law enforced by the Lagrange multiplier a_0. The longitudinal part, instead, represents the additional bare kinetic term for the vortices. Given that now the gauge field a_μ has acquired a kinetic term, we are also forced to introduce new integers M_μ via the Poisson formula to guarantee that the charge current $K_{\mu\nu} b_\nu$ remains an integer:

$$
\begin{aligned}
Z_{SD} &= \sum_{\substack{\{Q_\mu\}\\ \{M_\mu\}}} \int \mathcal{D}a_\mu \int \mathcal{D}b_\mu \exp(-S_{SD}) \ , \\
S_{SD} &= \sum_x -i2\pi \ a_\mu K_{\mu\nu} b_\nu + \frac{p_i^2}{2l_0 E_J} + \frac{\pi^2 q_i^2}{N^2 l_0 E_C} + i2\pi a_\mu Q_\mu + i2\pi b_\mu M_\mu \ , \qquad (31)
\end{aligned}
$$

where the new integers satisfy the constraint $\hat{\Delta}_\mu M_\mu = 0$ to guarantee gauge invariance. After a rescaling

$$
\begin{aligned}
A_0 &\equiv \frac{2\pi}{\sqrt{N} l_0} a_0 \ , \qquad & A_i \equiv \frac{2\pi}{\sqrt{N} l} a_i \ , \\
B_0 &\equiv \frac{2\pi}{\sqrt{N} l_0} b_0 \ , \qquad & B_i \equiv \frac{2\pi}{\sqrt{N} l} b_i \ , \qquad (32)
\end{aligned}
$$

we obtain finally

$$
\begin{aligned}
Z_{SD} &= \sum_{\substack{\{Q_\mu\}\\ \{M_\mu\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp(-S_{SD}) \ , \\
S_{SD} &= \sum_x \frac{l_0 l^2}{2e^2} \ F_i F_i - i \frac{l_0 l^2 \kappa}{2\pi} A_\mu k_{\mu\nu} B_\nu + \frac{l_0 l^2}{2g^2} \ f_i f_i \\
&\quad + i\sqrt{\kappa} \left(l_0 Q_0 A_0 + l Q_i A_i \right) + i\sqrt{\kappa} \left(l_0 M_0 B_0 + l M_i B_i \right) \ , \qquad (33)
\end{aligned}
$$

where F_i and f_i are defined in (14) and

$$
e^2 = 2NE_C \ , \qquad \kappa = N \ , \qquad g^2 = \frac{4\pi^2}{N} E_J \ . \qquad (34)
$$

This is exactly the partition function of our lattice Chern-Simons model (15) in the limit of infinite magnetic permeability $\eta = \infty$ and with additional, integer-valued link variables Q_μ and M_μ coupled to the two gauge fields. Note that, with the above identifications, the topological Chern-Simons mass (8) coincides with the Josephson plasma frequency:

$$
m = \sqrt{2N^2 E_C E_J} \ . \qquad (35)
$$

In the physical case $N = 2$ this reduces to $m = \sqrt{8E_C E_J}$. From the kinetic terms in (33) we can also read off the charge and vortex masses:

$$
\begin{aligned}
m_q &= \frac{1}{l^2 g^2} = \frac{N}{4\pi^2 l^2 E_J} \ , \\
m_\phi &= \frac{1}{l^2 e^2} = \frac{1}{2N l^2 E_C} \ . \qquad (36)
\end{aligned}
$$

In the regime $ml \leq O(1)$, which is typically experimentally relevant, we can choose $l_0 = l$: in this case the infinite magnetic permeability constitutes the only non-relativistic effect in the physics of Josephson junction arrays in the self-dual approximation. However, we expect this non-relativistic effect to be irrelevant as far as the phase structure and the charge-vorticity assignements are concerned. Therefore, for simplicity, we shall henceforth consider the relativistic model, by setting $l_0 = l$ and $\eta = 1$, although it is not hard to incorporate a generic value of η into our subsequent formalism:

$$\begin{aligned} Z_{SD} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp(-S_{SD}) \ , \\ S_{SD} &= \sum_x \frac{l^3}{2e^2} F_\mu F_\mu - i\frac{l^3\kappa}{2\pi} A_\mu k_{\mu\nu} B_\nu + \frac{l^3}{2g^2} f_\mu f_\mu + il\sqrt{\kappa} A_\mu Q_\mu + il\sqrt{\kappa} B_\mu M_\mu \end{aligned} \tag{37}$$

Josephson junction arrays in the self-dual approximation constitute thus a further, experimentally accessible example of the ideas presented in [17] and [18]. The action in (37) provides in fact a pure gauge theory representation of a model of interacting charges and vortices, represented by the conserved currents

$$\begin{aligned} q_\mu^{\rm charge} &\equiv \frac{\kappa^{\frac{3}{2}}}{2\pi}\, k_{\mu\nu} B_\nu \ , \\ \phi_\mu^{\rm vortex} &\equiv \frac{1}{2\pi\kappa^{\frac{1}{2}}}\, \hat{k}_{\mu\nu} A_\nu \ , \end{aligned} \tag{38}$$

where the prefactors are chosen so that the quantum of charge is given by κ, while the quantum of vorticity is given by $1/\kappa$ (factors of q_e and 2π are absorbed in the definitions of the gauge fields and the coupling constants).

In this framework, the mixed Chern-Simons term represents both the Lorentz force caused by vortices on charges (coupling of $q_\mu^{\rm charge}$ to the "electric" gauge field A_μ) and, by a summation by parts, the Magnus force caused by charges on vortices (coupling of $\phi_\mu^{\rm vortex}$ to the "magnetic" gauge field B_μ). The integer-valued link variables Q_μ and M_μ represent the (Euclidean) *topological excitations* [25] in the model. They satisfy the constraints

$$\begin{aligned} \hat{d}_\mu Q_\mu &= 0 \ , \\ \hat{d}_\mu M_\mu &= 0 \ . \end{aligned} \tag{39}$$

In a dilute phase they constitute closed electric (Q_μ) and magnetic (M_μ) loops on the lattice; in a dense phase there is the additional possibility of infinitely long strings. Due to the constraints (39) we can choose to represent these topological excitations as

$$\begin{aligned} Q_\mu &\equiv l k_{\mu\nu} Y_\nu \ , \qquad & Y_\nu \in Z \ , \\ M_\mu &\equiv l \hat{k}_{\mu\nu} X_\nu \ , \qquad & X_\mu \in Z \ , \end{aligned} \tag{40}$$

and reabsorb them in the mixed Chern-Simons term as follows:

$$S_{SD} = \sum_x \dots - i\frac{l^3\kappa}{2\pi} \left(A_\mu - \frac{2\pi}{l\sqrt{\kappa}} X_\mu \right) k_{\mu\nu} \left(B_\nu - \frac{2\pi}{l\sqrt{\kappa}} Y_\mu \right) + \dots \ . \tag{41}$$

In this representation it is clear that the topological excitations render the charge-vortex coupling *periodic* under the shifts

$$\begin{aligned} A_\mu &\quad \to A_\mu + \frac{2\pi}{l\sqrt{\kappa}}\, a_\mu \,, & a_\mu \in Z \,, \\ B_\mu &\quad \to B_\mu + \frac{2\pi}{l\sqrt{\kappa}}\, b_\mu \,, & b_\mu \in Z \,. \end{aligned} \tag{42}$$

In physical terms, the topological excitations implement the well-known [19] periodicity of the charge dynamics under the addition of an integer multiple of the flux quantum $1/\kappa$ per plaquette and the (less-known) periodicity of the vortex dynamics under the addition of an integer multiple of the charge quantum κ per site.

If we would require that the full action (including charge-charge and vortex-vortex interactions) (37) be periodic under the shifts (42) , then we would obtain the compact Chern-Simons model studied in [12] . In this case the relevant topological excitations would be essentially iX_μ and iY_μ: since these can also describe finite open strings, there is the additional possibility of electric and magnetic monopoles [25] . As we showed in [12], these monopoles play a crucial role in the regime $ml \ll 1$.

4 Phase structure analysis

In this section we investigate symmetry aspects and non-perturbative features of the model (37) due to the periodicity of the charge-vortex interactions encoded in the mixed Chern-Simons term. As expected, these depend entirely on the topological excitations which enforce the periodicity.

Upon a Gaussian integration the partition function (37) factorizes readily as

$$Z_{SD} = Z_{CS} \cdot Z_{\rm Top} \,, \tag{43}$$

where Z_{CS} is the pure gauge part defined in (15) and

$$\begin{aligned} Z_{\rm Top} &= \sum_{\substack{\{Q_\mu\} \\ \{M_\mu\}}} \exp\left(-S_{\rm Top}\right) \,, \\ S_{\rm Top} &= \sum_x \frac{e^2\kappa}{2l}\, Q_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2} Q_\nu + \frac{g^2\kappa}{2l}\, M_\mu \frac{\delta_{\mu\nu}}{m^2-\nabla^2} M_\nu \\ &\quad + i\frac{2\pi m^2}{l}\, Q_\mu \frac{k_{\mu\nu}}{\nabla^2\left(m^2-\nabla^2\right)} M_\nu \,, \end{aligned} \tag{44}$$

with m defined in (8), describes the contribution due to the topological excitations. The phase structure of our model is thus determined by the statistical mechanics of a coupled gas of closed or infinitely long electric and magnetic strings with short-range Yukawa interactions. The scale $(1/m)$ represents the width of these strings. In our case it is of the order of the lattice spacing l. The third term in the action (44), describing the topological

Aharonov-Bohm interaction of electric and magnetic strings, vanishes for strings separated by distances much bigger than $(1/m)$: in this case the denominator reduces to $m^2\nabla^2$ and, by using either one of the two equations in (40) and the constraints (39) one recognizes immediately that the whole term in the action reduces to ($i2\pi$ integer), which is equivalent to 0 (this reflects the fact that the original charges and vortices satisfy the Dirac quantization condition).

4.1 Free energy arguments

In order to establish the phase diagram of our model we use the free energy arguments for strings introduced in [26].

The usual argument for strings with Coulomb interactions [26] is that interactions between strings are unimportant for the phase structure because small strings interact via short-range dipole interactions, while large strings have most of their multipole moments canceled by fluctuations. This argument is even stronger in our case, where the interaction is anyway short-range. Therefore one retains only the self-energy of strings, which is proportional to their length, and phase transitions from dilute to dense phases appear when the entropy of large strings, also proportional to their length, overwhelms the self-energy. We shall also neglect the interaction term between electric and magnetic strings (imaginary term in the action (44)). This is clearly a good approximation if both types of topological excitations are dilute.

Thus, one assigns a free energy

$$F = \left(\frac{le^2\kappa}{2} G(ml)\, Q^2 + \frac{lg^2\kappa}{2} G(ml) M^2 - \mu \right) N \qquad (45)$$

to a string of length $L = lN$ carrying electric and magnetic quantum numbers Q and M, respectively. Here $G(ml)$ is the diagonal element of the lattice kernel $G(x-y)$ representing the inverse of the operator $l^2\,(m^2 - \nabla^2)$. Clearly $G(ml)$ is a function of the dimensionless parameter ml. The last term in (45) represents the entropy of the string: the parameter μ is given roughly by $\mu = \ln 5$, since at each step the string can choose between 5 different directions. In (45) we have neglected all subdominant functions of N, like a $\ln N$ correction to the entropy.

The condition for condensation of topological excitations is obtained by minimizing the free energy (45) as a function of N. If the coefficient of N in (45) is positive, the minimum is obtained for $N = 0$ and topological excitations are suppressed. If, instead, the same coefficient is negative, the minimum is obtained for $N = \infty$ and the system will favour the formation of large closed loops and infinitely long strings. Topological excitations with quantum numbers Q and M condense therefore if

$$\frac{le^2\kappa G(ml)}{2\mu}\, Q^2 + \frac{lg^2\kappa G(ml)}{2\mu}\, M^2 < 1\ . \qquad (46)$$

If two or more condensations are allowed by this condition one has to choose the one with the lowest free energy.

The condition (46) describes the interior of an ellipse with semi-axes $2\mu/(le^2\kappa G(ml))$ and $2\mu/(lg^2\kappa G(ml))$ on a square lattice of integer electric and magnetic charges. The phase diagram is obtained by investigating which points of the integer lattice lie inside the ellipse as its semi-axes are varied. We find it convenient to present the results in terms of the dimensionless parameters lm and e/g:

$$\frac{mlG(ml)\pi}{\mu} < 1 \qquad \rightarrow \begin{cases} \frac{e}{g} < 1 \ , & \text{electric condensation} \ , \\ \frac{e}{g} > 1 \ , & \text{magnetic condensation} \ , \end{cases}$$
$$\frac{mlG(ml)\pi}{\mu} > 1 \qquad \rightarrow \begin{cases} \frac{e}{g} < \frac{\mu}{mlG(ml)\pi} \ , & \text{electric condensation} \ , \\ \frac{\mu}{mlG(ml)\pi} < \frac{e}{g} < \frac{mlG(ml)\pi}{\mu} \ , & \text{no condensation} \ , \\ \frac{e}{g} > \frac{mlG(ml)\pi}{\mu} \ , & \text{magnetic condensation} \ . \end{cases} \tag{47}$$

As expected, these condensation patterns are symmetric around the the point $e/g = 1$, reflecting the self-duality of the model. In first approximation the electric (magnetic) condensation phase is characterized by the fact that $\{Q_\mu\}$ ($\{M_\mu\}$) fluctuate freely, while all $M_\mu = 0$ ($Q_\mu = 0$). Within this approximation it is clearly consistent to neglect altogether the interaction term between electric and magnetic strings in (45). Taking into account small loop corrections in the various phases can lead to a renormalization of coupling constants and masses and, correspondingly, to a shift of the critical couplings $(ml)_{\rm crit}$ and $(e/g)_{\rm crit}$ for the phase transitions. A notable exception is the case in which there is only one phase transition: in this case the critical coupling is $(e/g)_{\rm crit} = 1$ due to self-duality.

4.2 Wilson and 't Hooft loops

In order to distinguish the various phases we introduce the typical order parameters of lattice gauge theories [25], namely the *Wilson loop* for an electric charge q and the *'t Hooft loop* for a vortex ϕ:

$$\begin{aligned} L_W &\equiv \exp\left(i\frac{q}{\kappa^{\frac{1}{2}}}\sum_x lq_\mu A_\mu\right) \ , \\ L_H &\equiv \exp\left(i\phi\kappa^{\frac{3}{2}}\sum_x l\phi_\mu B_\mu\right) \ , \end{aligned} \tag{48}$$

where q_μ and ϕ_μ vanish everywhere but on the links of the closed loops, where they take the value 1. Since the loops are closed they satisfy

$$\hat{d}_\mu q_\mu = \hat{d}_\mu \phi_\mu = 0 \ . \tag{49}$$

The expectation values $\langle L_W \rangle$ and $\langle L_H \rangle$ can be used to characterize the various phases. First of all they measure the interaction potential between static, external test charges q and $-q$ and vortices ϕ and $-\phi$, respectively [25] . Secondly, by representing the closed loops q_μ and ϕ_μ as

$$\begin{aligned} q_\mu &\equiv lk_{\mu\nu}A^q_\nu \ , \\ \phi_\mu &\equiv l\hat{k}_{\mu\nu}A^\phi_\nu \ , \end{aligned} \tag{50}$$

we can rewrite the Wilson and ’t Hooft loops as

$$\begin{aligned} L_W &= \exp\left(i \frac{q}{\kappa^{\frac{1}{2}}} \sum_x l^2 A^q_\mu F_\mu \right) \ , \\ L_H &= \exp\left(i \kappa^{\frac{3}{2}} \phi \sum_x l^2 A^\phi_\mu f_\mu \right) \ , \end{aligned} \tag{51}$$

which is a lattice version of Stoke’s theorem, the integers A^q_μ and A^ϕ_μ $(= 0, \pm 1)$ representing the area elements of the surfaces spanned by the closed loops. The second terms of the expansions of $\langle L_W \rangle$ and $\langle L_H \rangle$ in powers of q and ϕ measure therefore the gauge invariant correlation functions $\langle F_\mu(x) F_\nu(y) \rangle$ and $\langle f_\mu(x) f_\nu(y) \rangle$. Third, if we represent ϕ_μ as

$$\phi \phi_\mu \equiv \frac{l^2}{2\pi} \ \hat{k}_{\mu\nu} A^{\rm e.m.}_\nu \ , \tag{52}$$

we can also rewrite the ’t Hooft loop as

$$L_H = \exp\left(i \sum_x l^3 A^{\rm e.m.}_\mu q^{\rm charge}_\mu \right) \ . \tag{53}$$

With the interpretation of $A^{\rm e.m.}_\mu$ as an external electromagnetic gauge potential the expectation value of the ’t Hooft loop measures the *electromagnetic response* of the system in the various phases. An analogous relation clearly holds for the Wilson loop.

The expectation values of the Wilson and ’t Hooft loops are easily obtained by combining the definitions 48 with 37:

$$\begin{aligned} \langle L_W \rangle &= \frac{Z_{\rm Top}\left(Q_\mu + \frac{q}{\kappa} q_\mu, M_\mu\right)}{Z_{\rm Top}\left(Q_\mu, M_\mu\right)} \ , \\ \langle L_H \rangle &= \frac{Z_{\rm Top}\left(Q_\mu, M_\mu + \phi\kappa\phi_\mu\right)}{Z_{\rm Top}\left(Q_\mu, M_\mu\right)} \ , \end{aligned} \tag{54}$$

where the notation is self-explanatory. In the following we shall analyze these expressions in the various phases obtained in (47). We shall mostly only indicate the form of small loop corrections: a full renormalization group analysis is beyond the scope of the present paper and we won’t be able to predict the orders of the phase transitions.

Let us begin with the *electric condensation phase*. In this phase the ground state contains many infinitely long electric strings Q_μ. These have a crucial effect on the gauge symmetry associated with the gauge field A_μ. To see this let us consider a gauge transformation $A_\mu \to A_\mu + d_\mu \Lambda$, where, for simplicity, we take Λ as a function of the component x^1 only. If we choose the usual boundary conditions $F_\mu = f_\mu = 0$ at infinity, the change of the action (37) under the above gauge transformation is given by

$$\Delta S_{SD} = \sum_{x^0, x^2} i \sqrt{\kappa} \left(\Lambda(x^1 = +\infty) Q_1(x^1 = +\infty) - \Lambda(x^1 = -\infty) Q_1(x^1 = -\infty) \right) \ . \tag{55}$$

In a dilute phase, with only small closed loops, $Q_1(x^1 = +\infty) = Q_1(x^1 = -\infty) = 0$ and the action is automatically gauge invariant. In a dense phase, with many infinitely long strings, $Q_1(x^1 = +\infty)$ and $Q_1(x^1 = -\infty)$ are generically different from zero. Gauge invariance requires that ΔS_{SD} vanishes modulo $i2\pi$. In the dense phase this is realized only if Λ takes the values

$$\Lambda = \frac{2\pi}{\sqrt{\kappa}} n \, , \qquad\qquad n \in Z \, , \tag{56}$$

at infinity. This means that, in the electric condensation phase, the *global* gauge symmetry is spontaneously broken down to the discrete gauge group Z, so that the total (global) symmetry of this phase is $Z_A \times R_B$.

The Wilson loop expectation value takes a particularly simple form if the external test charges are multiples of the charge quantum: $q = n\kappa$, $n \in Z$. In fact, since we sum over $\{Q_\mu\}$, the integer loop variables nq_μ can be absorbed by a redefinition of the appropriate Q_μ's, with the result

$$\langle L_W(q = n\kappa)\rangle = 1 \, . \tag{57}$$

This indicates that, in this phase, external test charges $q = n\kappa$ are perfectly *screened* by the topological excitations and behave thus freely. In order to compute the Wilson loop expectation value for generic q we have to perform explicitly the sum over $\{Q_\mu\}$. To this end we have to remember the constraint $\hat{d}_\mu Q_\mu = 0$. We solve this constraint by representing $Q_\mu = l k_{\mu\nu} n_\nu$ and summing over $\{n_\mu\}$, with the appropriate gauge fixing. We then use Poisson's formula 25 to turn this sum into an integral, by introducing a new set of integer link variables $\{k_\mu\}$ satisfying $\hat{d}_\mu k_\mu = 0$ in order to guarantee the gauge invariance under $n_\mu \to n_\mu + l d_\mu i$. At this point we can perform explicitly the Gaussian integration over $\{n_\mu\}$. In the approximation of neglecting terms proportional to ∇^2/m^2 (keeping such terms would not alter substantially the result) the new integers $\{k_\mu\}$ can be absorbed by a redefinition of the magnetic topological excitations $\{M_\mu\}$, giving the result:

$$\begin{aligned} \langle L_W \rangle \quad &= \frac{Z_{\mathrm{corr.}}\,(q_\mu)}{Z_{\mathrm{corr.}}\,(q_\mu = 0)} \, , \\ Z_{\mathrm{corr.}}\,(q_\mu) \quad &= \sum_{\{M_\mu\}\mathrm{loops}} \exp \sum_x \left(-\frac{g^2\kappa}{2l}\, M_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} M_\nu + i2\pi \frac{q}{\kappa} A^q_\mu M_\mu \right) \, . \end{aligned} \tag{58}$$

Since the magnetic topological excitations are in a dilute phase we have to sum only over small closed loops: in this phase the dominant part of $\ln\langle L_W\rangle$ vanishes for generic q and the whole result is given by small loop corrections. These are identical in form to the small loop corrections for the correlation functions in the low-temperature phase of the three-dimensional XY model; correspondingly the Wilson loop expectation value can be computed by exactly the same low-temperature expansion used for the XY model. The first-order term in this expansion is obtained by considering only the smallest possible lattice loops and gives the result

$$\langle L_W \rangle = \exp \left(2\mathrm{e}^{-\frac{g^2\kappa l}{6}} \sum_{x,\mu} \left[\cos \left(2\pi \frac{q}{\kappa} q_\mu \right) - 1 \right] \right) \, . \tag{59}$$

The periodicity of this result is a direct consequence of the spontaneous symmetry breaking $R_A \to Z_A$. This implies also that the gauge invariant correlation function reduces to

$$\langle F_\mu(x) F_\nu(y) \rangle \propto \left(\delta_{\mu\nu} \nabla^2 - d_\mu \hat{d}_\nu \right) \ \frac{\delta_{x,y}}{l^3} \ , \tag{60}$$

which is essentially a contact term on the scale of the lattice spacing.

The computation of the 't Hooft loop expectation value follows exactly the same lines as the above computation of the Wilson loop. The results is

$$\begin{aligned} \langle L_H \rangle \qquad &= \exp \left(-\frac{g^2 \kappa^3 \phi^2}{2l} \sum_x \phi_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} \phi_\nu \right) \ \frac{Z_{\rm corr}\left(\phi_\mu\right)}{Z_{\rm corr}\left(\phi_\mu = 0\right)} \ , \\ Z_{\rm corr}\left(\phi_\mu\right) \qquad &= \sum_{\{M_\mu\}{\rm loops}} \exp \left(-\frac{g^2 \kappa}{2l} \sum_x M_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} M_\nu + 2\kappa\phi \ M_\mu \frac{\delta_{\mu\nu}}{-\nabla^2} \phi_\nu \right) \ . \end{aligned} \tag{61}$$

The first few terms in the expansion of the small loop corrections can again be computed with the same techniques as in the low-temperature phase of the XY model. One finds that their contribution amounts to perturbative corrections of the Coulomb coupling constant $g^2 \kappa^3 \phi^2 / 2l$ of the dominant term in (61).

¿From (61) we can extract the nature of the electric condensation phase. First of all, by considering, as usual, a rectangular loop of length T in the imaginary time direction and of length R in one of the spatial directions and computing the dominant large-T behaviour of $\ln \langle L_H \rangle$ we find that the interaction potential between external test vortices of strength ϕ and $-\phi$ is proportional to $\ln R$. Vortices are thus logarithmically *confined*, which amounts to the *Meissner effect*. Secondly, by using the representations (50) and 51 , we find the correlation function

$$\langle f_\mu(x) f_\nu(y) \rangle \propto \frac{\delta_{\mu\nu} \nabla^2 - d_\mu \hat{d}_\nu}{\nabla^2} \ \frac{\delta_{x,y}}{l^3} \ , \tag{62}$$

which is long-range, indicating that the "B_μ-photon" is *massless*. This is the massless excitation associated with the spontaneous symmetry breaking of the global gauge symmetry $R_A \to Z_A$. Third, by using the representations (52) and (53), we find that the induced electromagnetic current is given by

$$J_\mu^{\rm e.m.} \propto \left(\delta_{\mu\nu} - \frac{d_\mu \hat{d}_\nu}{\nabla^2} \right) \ A_\nu^{\rm e.m.} \ , \tag{63}$$

which is the standard *London form*. We thus conclude that the electric condensation phase is actually a *superconducting phase*.

No further computation is needed to extract the nature of the *magnetic condensation phase*: this is the exact dual of the electric condensation phase just described. Specifically, the global gauge symmetry associated with B_μ is spontaneoulsy broken down to Z_B, so that the total symmetry of this phase is $R_A \times Z_B$. Correspondingly, the "A_μ-photon" is *massless* and the $\langle F_\mu(x) F_\nu(y) \rangle$ correlation function is long-range. Electric charges are logarithmically *confined*, which means that an infinite energy (voltage) is required to separate a neutral pair of charges. We call this phase with infinite resistance a *superinsulator*. In real Josephson junction arrays we expect however the conduction gap to be large but finite due to the small ground capacity C_0, resulting in a normal insulator.

If $mlG(ml)\pi/\mu > 1$ a third phase can open up between the superconducting and superinsulating phases. In this third phase both the electric and the magnetic topological excitations are dilute. Far away from the phase transitions and to first approximation we can neglect them altogether. This gives the result

$$\begin{aligned} \langle L_W \rangle \quad &= \exp\left(-\frac{e^2 q^2}{2l\kappa} \, q_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} q_\nu \right) \ , \\ \langle L_H \rangle \quad &= \exp\left(-\frac{g^2 \phi^2 \kappa^3}{2l} \, \phi_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} \phi_\nu \right) \ . \end{aligned} \tag{64}$$

Small loop corrections to these results can be obtained by restricting the $\{Q_\mu\}$ and $\{M_\mu\}$ sums in (54) to small closed loops and using again the same techniques as in the low-temperature expansion of the XY model. These will lead to perturbative corrections of the coupling constants and masses in (64); however the first-order result (64) is enough to establish the nature of this phase. The global symmetry characterizing this phase is $R_A \times R_B$ and, corrrespondingly, both "photons" are massive, resulting in short-range correlation functions $\langle F_\mu(x) F_\nu(y) \rangle$ and $\langle f_\mu(x) f_\nu(y) \rangle$. Both charges and vortices interact via short-range Yukawa potentials and behave thus freely when separated by distances larger then the scale $(1/m)$. In presence of any dissipation mechanism (which would not alter the other two phases) this third phase corresponds thus to a *metallic* phase of the Josephson junction array.

In conclusion we can represent the phase diagram of our model as follows:

$$\begin{aligned} \frac{mlG(ml)\pi}{\mu} < 1 \qquad &\to \begin{cases} \frac{e}{g} < 1 \ , & \text{superconductor } (Z_\times R_B) \ , \\ \frac{e}{g} > 1 \ , & \text{superinsulator } (R_A \times Z_B) \ , \end{cases} \\ \frac{mlG(ml)\pi}{\mu} > 1 \qquad &\to \begin{cases} \frac{e}{g} < \frac{\mu}{mlG(ml)\pi} \ , & \text{superconductor } (Z_A \times R_B) \ , \\ \frac{\mu}{mlG(ml)\pi} < \frac{e}{g} < \frac{mlG(ml)\pi}{\mu} \ , & \text{metal } (R_A \times R_B) \ , \\ \frac{e}{g} > \frac{mlG(ml)\pi}{\mu} \ , & \text{superinsulator } (R_A \times Z_B) \ , \end{cases} \end{aligned} \tag{65}$$

where we have indicated in parenthesis the global symmetries of the various phases.

A numerical computation of the function $mlG(ml)\pi/\mu$ for the value $\mu = \ln 5$, gives an indication that a window for the metallic phase is open for ml just larger than 1, while in the regime $ml \leq O(1)$, relevant for Josephson junction arrays, a single phase transition from a superconductor to a superinsulator at $(e/g) = 1$ is favoured.

The experimental results for Josephson junction arrays, which are essentially resistance measurements as a function of temperature in arrays with O(10^4) cells, shows, etrapolating at zero-temperature, a quantum phase transition between an insulator and a superconductor in the vicinity of the self-dual point $E_J/E_C = 2/\pi^2 \simeq 0.2$. This is in perfect agreement with $(e/g) = 1$. In fact, we have: $E_J/E_C = (2g^2)/(e^2\pi^2)$ and, for $(e/g) = 1$, we get $E_J/E_C = 2/\pi^2 \simeq 0.2$.

5 Superconductor with topological order

Topologically ordered superconductors have a long-distance hydrodynamic action which can be entirely formulated in terms of generalized compact gauge fields, the dominant term being the topological BF action.

BF theories are topological theories that can be defined on manifolds M_{d+1} of any dimension (here d is the number of spatial dimensions) and play a crucial role in models of two-dimensional gravity [8]. In [12] we have shown that the BF term also plays a crucial role in the physics of Josephson junction arrays.

The BF term [7] is the wedge product of a p-form B and the curvature dA of a (d-p) form A:

$$S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} B_p \wedge dA_{d-p} \ , \tag{66}$$

where k is a dimensionless coupling constant. This can also be written as

$$S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} A_{d-p} \wedge dB_p \ . \tag{67}$$

The integration by parts does not imply any surface term since we will concentrate on compact spatial manifolds without boundaries and we require that the fields go to pure gauge configuration at infinity in the time direction. Indeed this action has a generalized Abelian gauge symmetry under the transformation

$$B \to B + \eta \ , \tag{68}$$

where η is a closed p form: $d\eta = 0$. Gauge transformations:

$$A \to A + \xi \ , \tag{69}$$

with ξ a closed (d-p) form instead, change the action by a surface term. This, however vanishes with the boundary conditions we have chosen.

Here we will be interested in the special case where A_1 is a 1-form and, correspondingly, B_{d-1} is a (d-1)-form:

$$S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} A_1 \wedge dB_{d-1} \ . \tag{70}$$

In the special case of (3+1) dimensions, B is the well-known Kalb-Ramond tensor field $B_{\mu\nu}$ [27].

In the application to superconductivity, the conserved current $j_1 = *dB_{d-1}$ represents the charge fluctuations, while the generalized current $j_{d-1} = *dA_1$ describes the conserved fluctuations of (d-2)-dimensional vortex lines. As a consequence, the form B_{d-1} must be considered as a pseudo-tensor, while A_1 is a vector, as usual. The BF coupling is thus P- and T-invariant.

The low-energy effective theory of the superconductor can be entirely expressed in terms of the generalized gauge fields A_1 and B_{d-1}. The dominant term at long distances is the BF term; the next terms in the derivative expansion of the effective theory are the kinetic terms for the two gauge fields (for simplicity of presentation we shall assume relativistic invariance), giving:

$$\begin{aligned} S_{TM} \quad &= \int_{M_{d+1}} \tfrac{-1}{2e^2} dA_1 \wedge * dA_1 + \tfrac{k}{2\pi} A_1 \wedge dB_{d-1} \\ &\qquad + \tfrac{(-1)^{d-1}}{2g^2} dB_{d-1} \wedge * dB_{d-1} \ , \end{aligned} \tag{71}$$

where e^2 and g^2 are coupling constants of dimension m^{-d+3} and m^{d-1} respectively.

The BF-term is the generalization to any number of dimensions of the Chern-Simons mechanism for the topological mass [22]. To see this let us now compute the equation of motion for the two forms A and B:

$$\frac{1}{g^2} d * dB_{d-1} = \frac{k}{2\pi} dA_1 \ , \tag{72}$$

and

$$\frac{1}{e^2} d * dA_1 = \frac{k}{2\pi} dB_{d-1} \ . \tag{73}$$

Applying $d*$ on both sides of (72) and (73) we obtain

$$\begin{aligned} d * d * dA_1 - \tfrac{ke^2}{2\pi} d * dB_{d-1} = 0 \ , \\ d * d * dB_{d-1} - \tfrac{kg^2}{2\pi} d * dA_1 = 0 \ . \end{aligned} \tag{74}$$

The expression $*d*$ is proportional to δ, the adjoint of the exterior derivative [28]. Substituing $d * dB_{d-1}$ and $d * dA_1$ in (74) with the expression coming from (72) and (73) we obtain

$$\begin{aligned} \left(\Delta + m^2\right) dA_1 = 0 \ , \\ \left(\Delta + m^2\right) dB_{d-1} = 0 \ , \end{aligned} \tag{75}$$

where $\Delta = d\delta$ (when acting on an exact form) and $m = \frac{keg}{2\pi}$ is the topological mass. This topological mass plays the role of the gap characterizing the superconducting ground state. Note that the gap arises here from a topological mechanism and not from a local order parameter acquiring a vacuum expectation value. Equations (72) and (73) tell us that charges are sources for vortex line currents encircling them and viceversa. This is the coupling between charges and vortices at the origin of the gap. A related mechanism for topological mass generation in (3+1)-dimensional gauge theories is the generalization of the Schwinger mechanism proposed in [29].

Let us now consider the special case of (2+1) dimensions (d=2). In this case also B becomes a (pseudo-vector) 1-form and, correspondingly the BF term reduces to a mixed Chern-Simons term. This can be diagonalized by a transformation $A = \frac{a+b}{2}$, $B = a - b$, giving

$$S_{BF}(d=2) = \frac{k}{4\pi} \int a \wedge da - \frac{k}{4\pi} \int b \wedge db \ . \qquad (76)$$

The result is a doubled Chern-Simons model for gauge fields of opposite chirality. This action, including its non-Abelian generalization with kinetic terms was first considered in [30]. It is the simplest example of the class of P- and T-invariant topological phases of strongly correlated (2+1)-dimensional electron systems considered in [10]. Indeed, the BF term is the natural generalization of such doubled Chern-Simons models to any dimension. Doubled (or mixed) Chern-Simons models are thus particular examples in two spatial dimensions of a wider class of P- and T-invariant topological fluids that have a superconducting phase. These fluids are described by the topological BF theory with compact support for both gauge fields.

Topological BF models provide also a generalization of anyons to arbitrary dimensions. While in (2+1) dimensions fractional statistics arises from the representations of the braid group, encoding the exchange of particles, in (3+1) dimensions it arises from the adiabatic transport of particles around vortex strings and, in (d+1) dimensions, from the motion of an hypersurface Σ_h around another hypersurface Σ_{d-h}. The relevant group in this case is the motion group and the statistical parameter is given by $\frac{2\pi}{k} h(d-h)$, where k is the BF coupling constant [31].

Let us now illustrate the mechanism of superconductivity. To this end we shall from now on consider only rational $k = \frac{k_1}{k_2}$ with k_i integers, and specialize to manifolds $M_{d+1} = M_d \times R_1$, whith R_1 representing the time direction.

The compactness of the gauge fields allows for the presence of topological defects, both electric and magnetic. The electric topological defects couple to the form A_1 and are string-like objects described by a singular closed 1-form Q_1. Magnetic topological defects couple to the form B_{d-1} and are closed (d-1)-branes described by a singular (d-1)-dimensional form Ω_{d-1}. These forms represent the singular parts of the field strenghts dA_1 and dB_{d-1}, allowed by the compactness of the gauge symmetries [25], and are such that the integral of their Hodge dual over any hypersurface of dimensions d and 2, respectively, is 2π times an integer as can be easily derived using a lattice regularization. Contrary to the currents j_1 and j_{d-1}, which represent charge- and vortex-density waves, the topological defects describe localized charges and vortices. In the effective theory these have structure on the scale of the ultraviolet cutoff.

We will not discuss here the conditions for the condensation of topological defects, but we will show, instead that the phase of electric condensation describes a superconducting phase in any dimension. A detailed analysis would require the use of an ultraviolet regularization. Here we will present a formal derivation implying the ultraviolet regularization (e.g. a lattice regularization).

In the phase in which electric topological defects condense (while magnetic ones are dilute) the partition function requires a formal sum also over the form Q_1

$$Z = \int \mathcal{D}A\mathcal{D}B\mathcal{D}Q$$
$$\exp\left[i\frac{k}{2\pi}\int_{M_{d+1}} \left(A_1 \wedge dB_{d-1} + A_1 \wedge *Q_1 \right) \right] \ . \tag{77}$$

Let us now compute the expectation value of the 't Hooft operator, $\langle L_H \rangle$, which represents the amplitude for creating and separating a pair of vortices with fluxes $\pm\phi$:

$$\langle L_H \rangle = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}B\mathcal{D}Q$$
$$\exp\left[i\frac{k}{2\pi}\int_{M_{d+1}} \left(A_1 \wedge dB_{d-1} + A_1 \wedge *Q_1 \right) + i\frac{k}{2\pi}\phi\int_{S_{d-1}} B_{d-1} \right] \ . \tag{78}$$

Using Stokes' theorem we can rewrite

$$\int_{S_{d-1}} B_{d-1} = \int_{S_d} dB_{d-1} \ , \tag{79}$$

where the surface S_d is such that $\partial S_d \equiv S_{d-1}$ and represents a compact orientable surface on M_d. Inserting (79) in (78) and integrating over the field A we obtain:

$$\langle L_H \rangle \ \propto \int \mathcal{D}B\mathcal{D}Q\ \delta\left(dB_{d-1} + *Q_1 \right)$$
$$\exp\left[i\frac{k}{2\pi}\phi\int_{S_d} dB_{d-1} \right] \ . \tag{80}$$

Integrating over B gives then:

$$\langle L_H \rangle = \propto \int \mathcal{D}Q\ \exp\left[-i\frac{k}{2\pi}\phi\int_{S_d} *Q_1 \right] \ . \tag{81}$$

The Poisson summation formula implies finally that the 't Hooft loop expectation value vanishes for all flux strengths ϕ different from

$$\frac{\phi}{k_2} = \frac{2\pi}{k_1}\, n \qquad\qquad n \in N \ . \tag{82}$$

This is nothing else than the Meissner effect, illustrating that the electric condensation phase is superconducting. Indeed, the electric condensate carries k_1 fundamental charges of unit $1/k_2$ as is evident from (77), and correspondingly vortices must carry an integer multiple of the fundamental fluxon $2\pi/(k_1/k_2)$. All other vorticities are confined: in this purely topological long-distance theory the confining force is infinite; including the higher order kinetic terms (71) and the UV cutoff one would recover a generalized area law.

Another way to see this is to compute the current induced by an external electromagnetic field $A_{\rm ext}$. The corresponding coupling is $\int_{M_{d+1}} A_{\rm ext} \wedge \left(*j_1 + *Q_1 \right) \propto$

$\int_{M_{d+1}} A_{\rm ext} \wedge (dB_{d-1} + *Q_1)$. Since $A_{\rm ext}$ can be entirely reabsorbed in a redefinition of the gauge field A_1, the induced current vanishes identically, $j_{\rm ind} = 0$. This is just the London equation in the limit of zero penetration depth. Including the higher-order kinetic terms for the gauge fields and the UV cutoff one would again recover the standard form of the London equation.

Associated with the confinement of vortices there is a breakdown of the original U(1) matter symmetry under transformations $A_1 \to A_1 + d\lambda$. To see this let us consider the effect of such a transformation on the partition function (77) with an electric condensate. Upon integration by parts, the exponential of the action acquires a multiplicative factor

$$\exp i \frac{k_1}{2\pi k_2} \left(\int_{M_d, t=+\infty} \lambda \wedge *Q_1 - \int_{M_d, t=-\infty} \lambda \wedge *Q_1 \right) . \tag{83}$$

Assuming a constant λ, we see that the only values for which the partition function remains invariant are

$$\lambda = 2\pi \, n \, \frac{k_2}{k_1} , \qquad n = 1 \dots k_1 , \tag{84}$$

which shows that the global symmetry is broken from U(1) to Z_{k_1}. Note that this is not the usual Landau mechanism ofspontaneous symmetry breaking. Indeed, there is no local order parameter and the order is characterized rather by the expectation value of non-local, topological operators.

The hallmark of topological order is the degeneracy of the ground state on manifolds with non-trivial topology as shown by Wen [32]. In (2+1) dimensions the degeneracy for the mixed Chern-Simons term was proven in [33] for the case of integer coefficient k of the Chern-Simons term.

The degeneracy of the ground state of the BF theory on a manifold with non-trivial topology was proven in [9] in (3+1) dimensions. This result can be generalized to compact topological BF models in any number of dimensions [31]. Consider the model (67) with $k = \frac{k_1}{k_2}$ on a manifold $M_d \times R_1$, with M_d a compact, path-connected , orientable d-dimensional manifold without boundaries. The degeneracy of the ground state is expressed in terms of the intersection matrix M_{mn} [34] with $m, n = 1....N_p$ and N_p the rank of the matrix, between p-cycles and (d-p)-cycles. N_p corresponds to the number of generators of the two homology groups $H_p(M_d)$ and $H_{d-p}(M_d)$ and is essentially the number of non-trivial cycles on the manifold M_d. The degeneracy of the ground state is given by $|k_1 k_2 M|^{N_p}$, where M is the integer-valued determinant of the linking matrix. In our case p = (d-1) and the degeneracy reduces to

$$|k_1 k_2 M|^{N_{d-1}} . \tag{85}$$

6 Frustration

The gauge theory formulation of JJA [12] clearly shows that the superconducting ground state is a P- and T- invariant generalization of Laughlin's incompressible quantum fluid.

The simplest example of a topological fluid [18] is a ground state described by a low energy effective action given solely by the topological Chern-Simons term

$$S = k/4\pi \int d^3x \; A_\mu \epsilon^{\mu\nu\alpha} \partial_\nu A_\alpha$$

for a compact $U(1)$ gauge field A_μ, whose dual field strength $F^\mu = \epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha$ yields the conserved matter current. In this case the degeneracy of the ground state on a manifold of genus g will be k^g (or $(k_1 k_2)^g$ if $k = k_1/k_2$ is a rational number). For planar unfrustrated JJA one finds that the topological fluid is described by two $k = 1$ Chern-Simons gauge fields of opposite chirality and, thus, there is no degeneracy of the ground state [12], [21].

In [21] we argued that frustrated JJA may support a topologically ordered ground state with non-trivial degeneracy on higher genus surfaces, described by a pertinent superconducting quantum fluid, thus providing a more interesting example of a system in which superconductivity arises from the topological mechanism proposed in [1] rather than from the usual Landau-Ginzburg mechanism. The role of frustration in the determination of the ground state degeneracy of JJA (in the limit $E_J \gg E_C$) was also analyzed in [35].

The fact that frustration can change the coefficient of the Chern-Simons term can be easily understood by the following example. Let us briefly review the mean field approach to spin liquid states [36]. This description is based on the Hubbard model that, following Anderson, is thought to be relevant for high T_c superconductivity. At half filling this model reduces to the Heisenberg model. Although very little is known rigourosly about the Heisenberg model on a square lattice, it is believed that the Heisenberg antiferromagnet is Néel ordered at T = 0. Is it possible to drive the model toward a disordered ground state to give rise to a spin liquid? The answer to this question is yes, and the key element to reach disorder is frustration. An example is to consider next to nearest neighbor interactions in the Heisenberg model: the result is a chiral spin liquid that breaks P- and T-invariance [37]. We will thus concentrate on the Heisenberg model, taking the point of view that the various spin liquid phases can be realized with small modification of its Hamiltonian [19]. Moreover our analysis can be exactly repeted for the Heisenberg model with next to nearest neighbour interactions presented in [37].

We will study a mean field theory for the Heisenberg antiferromagnet originally proposed in [36]. To this end we consider a system of spin $S = 1/2$ on a square lattice with nearest neighbor interactions and periodic boundary conditions. The Hamiltonian that describes this model is:

$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S_i} \mathbf{S_j} \; . \tag{86}$$

To obtain the mean field ground state for the spin liquid we use a fermion representation of the spin operator:

$$\mathbf{S_i} = \frac{1}{2} {c^+}_{i\alpha} \sigma_{\alpha\beta} c_{i\beta} \; , \tag{87}$$

where σ are the Pauli matrices. The operators (87) reproduce the spin 1/2 algebra of only if ${c^+}_{i\alpha} c_{i\alpha} = 1$, namely if there is only one fermion per site (half filling). Substituting (87) in

(86) we obtain (up to an additive constant):

$$H = \frac{1}{4} \sum_{\langle ij \rangle} J_{ij} {c^+}_{i\alpha} c_{j\alpha} {c^+}_{j\beta} c_{i\beta} \ . \tag{88}$$

Introducing the Hubbard-Stratonovich field χ_{ij} we can rewrite the four-fermion interaction as (we will concentrate here only on this term):

$$H = ... - \sum_{\langle ij \rangle} {c^+}_{i\alpha} \chi_{ij} c_{j\alpha} \ . \tag{89}$$

We will now treat the field χ_{ij} within the mean field approximation $\chi_{ij}^{\rm MF}$ and, since it is complex, we will parametrize it with an amplitude ρ_{ij} and a phase A_{ij}. Self consistency implies $\chi_{ij}^{\rm MF} = \langle {c^+}_{i\alpha} c_{j\alpha} \rangle$.

Various possible mean-field solutions have been studied. We will be interested in the case in which $\chi_{ij}^{\rm MF}$ generates a flux: the fermions behave as though they would be moving in a magnetic field and the P- and T-symmetries are broken; moreover, when an integer number of Landau levels is filled, the electron gas is incompressible. For the case in which the flux per plaquette is π, (half of quantum of flux), it has been shown that, after integrating out the fermion fields, the effective theory contains a Chern-Simons term with a coefficient $k = 2$ [37](note that in the construction presented in [37] in a certain range of parameter P- and T-symmetry are broken even with a flux π per plaquette). This corresponds to the case in which the first Landau level is completely filled. It also been shown that for generalized flux phases, where the flux per plaquette is $2\pi p/q$ with q an even integer, the effective theory contains a Chern-Simons term with a coefficient $k = q$ (for $q = 2$ we obtain the previous case, namely 1/2 quantum flux per plaquette). In this case we have $q/2$ Landau levels that are filled.

Let us now consider the hamiltonian:

$$H = \sum_{\langle ij \rangle} J_{ij} S_i U_{ij} S_j \ , \tag{90}$$

where U_{ij} is a frustration field that satisfies : $\prod_p U_{ij} = \exp 2i\pi f$. Here f is the frustration parameter. It acts as a fictitious magnetic field, and, due to the invariance of the theory under $f \to f + n, n \in Z$ and $f \to -f$ we have $0 \leq f \leq \frac{1}{2}$. Moreover since we have imposed periodic boundary conditions the flux must be a rational multiple of the quantum flux: $f = 2\pi p/q$. Let us follow the same line of reasoning as before. To this end with start with a flux π per plaquette (first Landau level filled), coming from the mean field approximation of (89). In this case we end up with a flux: $2\pi(1/2 + p/q) = 2\pi(q + 2p)/2q$ per plaquette. We will end up with the problem of the hopping of fermions in a fictitius magnetic field that is changed from $\pi \to 2\pi(q + 2p)/2q$, with q Landau levels filled. This frustrated model will be described by a Chern-Simons theory with coefficient $k = 2q$ while the unfrustrated model corresponds to a Chern-Simons theory with coefficient $k = 2$.

Let us now go back to JJA. It has been suggested that, in presence of n_q offset charge quanta per site and n_ϕ external magnetic flux quanta per plaquette in specific ratios, Josephson junction arrays might have incompressible quantum fluid [2] phases corresponding to purely two-dimensional *quantum Hall phases* for either charges [38] or vortices [14, 39].

In [12] we have shown that, if quantum Hall phases for charges or vortices are realized, then JJA naturally support a topologically ordered ground state and a phase in which they behave as a topological superconductor [1]; there is, in fact, a renormalization of the Chern-Simons coefficient yielding a non-trivial ground state degeneracy on the torus (and in general on manifolds with non-trivial topology). To implement a torus topology we impose doubly periodic conditions at the boundary of the square lattice.

To study this degeneracy let us consider the low energy limit of the partition function (31). For the following analysis we will take the time as a continuos parameter (we remeber that the discretization of time is allowed by the fact that charges are quantized, but is not a necessary condition). In this limit we have:

$$\begin{aligned} Z &= \sum_{\substack{\{Q_0\}\\ \{M_0\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp(-S) \ , \\ S &= \int dt \sum_{\mathbf{x}} -i\frac{1}{2\pi}\ A_\mu K_{\mu\nu} B_\nu + iA_0 Q_0 + iB_0 M_0 \ . \end{aligned} \tag{91}$$

This form of the partition function holds true also with toroidal boundary conditions. With continuos time the operator $K_{\mu\nu}$ is defined by $K_{00} = 0$, $K_{0i} = -\epsilon_{ij} d_j$, $K_{i0} = S_i \epsilon_{ij} d_j$ and $K_{ij} = -S_i \epsilon_{ij} \partial_0$, in terms of forward (backward) shift and difference operators S_i ($\hat{S}_i$) and d_i ($\hat{d}_i$). Its conjugate $\hat{K}_{\mu\nu}$ is defined by $\hat{K}_{00} = 0$, $\hat{K}_{0i} = -\hat{S}_i \epsilon_{ij} \hat{d}_j$, $\hat{K}_{i0} = \epsilon_{ij} \hat{d}_j$ and $\hat{K}_{ij} = -\hat{S}_j \epsilon_{ij} \partial_0$.

The topological excitations are described by the integer-valued fields Q_0 and M_0 and represent unit charges and vortices rendering the gauge field components A_0 and B_0 integers via the Poisson summation formula; their fluctuations determine the phase diagram [12]. In the classical limit the magnetic excitations are dilute and the charge excitations condense rendering the system a superconductor: vortex confinement amounts here to the Meissner effect. In the quantum limit, the magnetic excitations condense while the charged ones become dilute: the system exhibits insulating behavior due to vortex superconductivity accompanied by a charge Meissner effect.

By rewriting the topological excitations as the curl of an integer-valued field

$$\begin{aligned} Q_0 &\equiv K_{0i} Y_i \ , \qquad && Y_i \in Z \ , \\ M_0 &\equiv \hat{K}_{0i} X_i \ , \qquad && X_i \in Z \ , \end{aligned} \tag{92}$$

we get the mixed Chern-Simons term as follows:

$$\begin{aligned} Z &= \textstyle\sum_{\{X_i, Y_i\}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp\left(-S\right) \ , \\ S &= -\tfrac{1}{2\pi} i \textstyle\int dt \sum_{\mathbf{x}} A_0 K_{0i} \left(B_i - 2\pi Y_i\right) \\ &\quad + B_0 \hat{K}_{0i} \left(A_i - 2\pi X_i\right) + A_i K_{ij} B_j \ . \end{aligned} \tag{93}$$

¿From (93) one sees that the gauge field components A_i and B_i are angular variables due to their invariance under time-independent integer shifts. Such shifts do not affect the last term in the action, which contains a time derivative, and may be reabsorbed in the topological excitations X_i and Y_i, leaving also the first term of the action invariant. The low energy theory is thus compact.

In analogy with the conventional quantum Hall setting one should expect the charge and vortex transport properties to depend on the ratios of the offset charges (i.e. the filling fractions) (n_q/n_ϕ) and (n_ϕ/n_q), respectively. Due to the periodicity of the charge-vortex coupling, however, n_ϕ (n_q) is defined only modulo an integer as far as charge (vortex) transport properties are concerned. Using this freedom one may define effective filling fractions (we shall assume $n_q \geq 0$, $n_\phi \geq 0$ for simplicity) as

$$\begin{aligned} \nu_q &\equiv \frac{n_q}{n_\phi - \left[n_\phi\right]^- + \left[n_q\right]^+} \ , && 0 \leq \nu_q \leq 1 \ , \\ \nu_\phi &\equiv \frac{n_\phi}{n_q - \left[n_q\right]^- + \left[n_\phi\right]^+} \ , && 0 \leq \nu_\phi \leq 1 \ , \end{aligned} \tag{94}$$

where $[n_q]^{\pm}$ indicate the smallest (greatest) integer greater (smaller) than n_q. Of course, these effective filling fractions are always smaller than 1.

In [12] we assumed the existence of these quantum Hall phases and discussed them in the framework of the gauge theory representation of Josephson junction arrays, showing that, depending on certain parameters of the array there are both a charge quantum Hall phase and a vortex quantum Hall phase. Here we will concentrate on the low energy limit of the charge quantum Hall phase and we will show that the system has topological order and behaves as a superconductor when charge condenses.

The pertinent low energy theory is now given by:

$$S = \int dt \sum_{\mathbf{x}} -\frac{i}{\pi} A_\mu K_{\mu\nu} B_\nu - \frac{i\nu_q}{\pi} A_\mu K_{\mu\nu} A_\nu \ , \tag{95}$$

with $\nu_q = p/n$. The main difference with (91) is the addition of a pure Chern-Simons term for the A_μ gauge field . We have also rescaled the coefficient of the mixed Chern-Simons coupling by a factor of 2 (compare with (91)). This factor of 2 is a well-known aspect of Chern-Simons gauge theories [40]. Moreover, since in JJA the charge degrees of freedom are bosons, the allowed [12] filling fractions are given by $\nu_q = \frac{p}{n}$, with $pn = \text{even integer}$ in accordance with [41]. As a result, the action (95) may now be written in terms of two independent gauge fields A_μ and $B^q_\mu = B_\mu + \nu_q A_\mu$ yielding:

$$S = \int dt \sum_{\mathbf{x}} -\frac{i}{\pi} A_\mu K_{\mu\nu} B^q_\nu \ . \tag{96}$$

In describing JJA one has to require the periodicity of charge-vortex couplings; the coupling of the topological excitations enforcing the periodicity of the mixed Chern-Simons term $A_\mu K_{\mu\nu} B^q_\nu$ is then:

$$S = \int dt \sum_{x} \ldots + ipA_0 Q_0 + inB_0 M_0 \ , \tag{97}$$

that can be rewritten as:

$$S = \sum_{x} \ldots + ilpA_0 \left(Q_0 + M_0 \right) + ilnB_0^q M_0 \ . \tag{98}$$

Due to the replacement $B_\mu \to B_\mu^q$, the periodicities of the two original gauge fields are

$$\begin{aligned} A_i &\to A_i + \pi n \, a_i \ , & a_i \in Z \ , \\ B_i &\to B_i + \pi p \, b_i \ , & b_i \in Z \ , \end{aligned} \tag{99}$$

and

$$B_i^q \to B_i^q + \pi p \, b_i \ , \qquad b_i \in Z \ . \tag{100}$$

The resulting low energy theory is thus, again, compact.

Using the representation (92), one may rewrite the mixed Chern-Simons term as

$$\begin{aligned} S = \int dt \sum_{x} \ldots \quad & -i \tfrac{(pq/2)}{2\pi} \left(\tfrac{2A_0}{n} \right) K_{0i} \left(\tfrac{2B_i^q}{p} - 2\pi Y_i \right) \\ & -i \tfrac{(pq/2)}{2\pi} \tfrac{2B_0^q}{n} K_{0i} \left(\tfrac{2A_i}{n} - 2\pi X_i \right) \ . \end{aligned} \tag{101}$$

In this representation it is clear that the topological excitations render the charge-vortex coupling periodic under the shifts

$$\begin{aligned} A_i^{'} = \frac{2A_i}{n} &\to A_i^{'} + 2\pi \, a_i \ , & a_i \in Z \ , \\ B_i^{'} = \frac{2B_i^q}{p} &\to B_i^{'} + 2\pi \, b_i \ , & b_i \in Z \ . \end{aligned} \tag{102}$$

This model corresponds to two Chern-Simons terms with coefficients $\pm k/4\pi$ with $k = np/2$ an integer. It is worth to point out that, since B_ν^q does not have a definite parity (is a liner combination of a vector and a pseudovector) the model is not P- and T-invariant, as it must be due to the presence of the Chern-Simons term for the field A_μ.

The hallmark of topological order is the degeneracy of the ground state on manifolds with non-trivial topology [3]. The torus degeneracy on the lattice of the Chern-Simons model was computed in [42]. For a single Chern-Simons term this degeneracy is $(k)^g$ where k is the integer coefficient of the Chern-Simons term, and g the genus of the surface. In our case this degeneracy is $2 \times (k)^g = 2 \times \frac{np}{2}$, since we have two Chern-Simons terms. This degeneracy is exactly what is expected for a doubled Chern-Simons model [10], for which the physical Hilbert space is the direct product of the two Hilbert spaces of the component models.

We will now demonstrate that the phase where topological excitations Q_0 condense while M_0 are dilute describes an effective gauge theory of a superconducting state. The partition function is:

$$\begin{aligned} Z_{LE} &= \sum_{\{Q_0\}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu^q \exp\left(-S\right) \ , \\ S &= \int dt \sum_{\mathbf{x}} -\frac{ik}{2\pi} \, A_\mu K_{\mu\nu} B_\nu^q + \frac{ik}{2\pi} A_0 (2\pi Q_0) \ . \end{aligned} \tag{103}$$

To this end note first that a unit external charge, represented by an additional term $i2\pi a_0(t,\mathbf{x})\delta_{\mathbf{x}\mathbf{x_0}}$ is completely screened by the charge condensate, since it can be absorbed into a redefinition of the topological excitations Q_0. In order to characterize the superconducting phase we introduce the typical order parameter namely the 't Hooft loop of length T in the time direction:

$$L_H \equiv \exp\left(i\phi\frac{\kappa}{2\pi}\int dt \sum_x \phi_\mu B_\mu\right) \ , \tag{104}$$

where $\phi_0(t,\mathbf{x})$ = $(\theta\left(t+T/2\right)-\theta\left(t-T/2\right))\,\delta_{\mathbf{x}\mathbf{x_1}}$ - $(\theta\left(t+T/2\right)-\theta\left(t-T/2\right))\,\delta_{\mathbf{x}\mathbf{x_2}}$ and $\phi_i(-T/2,\mathbf{x})$, $\phi_i(T/2,\mathbf{x})$ are unit links joining $\mathbf{x_1}$ to $\mathbf{x_2}$ and $\mathbf{x_2}$ to $\mathbf{x_1}$ at fixed time and vanishing everywhere else. Its vacuum expectation value $\langle L_H \rangle$ yields the amplitude for creating a separated vortex-antivortex pair of flux ϕ, which propagates for a time T and is then annihilated in the vacuum.

Since we replaced $B_\mu \to B_\mu^q$, we may rewrite the 't Hooft loop as:

$$L_H \equiv \exp\left(i\phi\frac{\kappa}{2\pi}\int dt \sum_x \left(\phi_\mu B_\mu^q - \frac{p}{n}A_\mu\phi_\mu\right)\right) \ . \tag{105}$$

To compute $\langle L_H \rangle$ one should integrate first over the gauge field B_μ^q to get

$$\begin{aligned}\langle L_H \rangle \qquad & \propto \sum_{\{Q_0\}} \int \mathcal{D}A_\mu \, \delta\left(\hat{K}_{\mu\nu}A_\nu - \phi\phi_\mu\right) \times \\ & \exp i\phi\frac{\kappa}{2\pi}\left(\int dt \sum_{\mathbf{x}} A_0(2\pi Q_0) - \frac{p}{n}A_\mu\phi_\mu\right) \ . \end{aligned} \tag{106}$$

The sum over Q_0 enforces the condition that $\frac{k}{2\pi}A_0$ be an integer. As a consequence, $\hat{K}_{i0}A_0 = \frac{2\pi}{k}n_i = \phi\phi_i$ with n_i an integer. We thus have:

$$\phi = \frac{2\pi}{k}q, q \in Z \ ; \tag{107}$$

thus, $\langle L_H \rangle$ vanishes for all fluxes different from an integer multiple of the fundamental fluxon, which is just the Meissner effect. In the low-energy effective gauge theory vortex-antivortex pairs are confined by an infinite force which becomes logarithmic upon including also higher-order Maxwell terms.

¿From (101) one has (depending if p is an even integer or n is an even integer) either that n is the charge unit with $p/2$ units of charge, or, viceversa, that p is the charge unit with $n/2$ units of charge. By rewriting (107) as $\phi = \frac{2}{pn}2\pi n$ one finds the standard flux quantization $\phi = \frac{2\pi}{Ne}n$ where e is the charge unit and N is the number of units of charge.

In this paper we have derived a superconductivity mechanism which is not based on the usual Landau theory of spontaneous symmetry breaking. Our considerations here focused on the low-energy effective theory in order to expose the physical basis of the topological superconducivity mechanism. It is however crucial to stress that the simplest example (k=1) of this type of topological superconductivity is concretely realized as the global superconductivity mechanism in planar Josephson junction arrays, as we have shown in [12]. Naturally, it would be most interesting to find examples of microscopic models realizing this superconductivity mechanism with more complex degeneracy patterns. A possibilty is represented by models with frustration. The frustrated planar JJA we have analized provide, in the quantum Hall phases, an explicit example of both topological order with non-trivial ground state degeneracy on manifolds with non-trivial topology, and of a new superconducting behavior [21] analogous to Laughlin's quantum Hall fluids.

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